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# ALMOST RICCI SOLITONS AND $K$-CONTACT GEOMETRY

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**Abstract:** We give a short Lie-derivative theoretic proof of the following recent result of Barros et al. "A compact non-trivial almost Ricci soliton with constant scalar curvature is gradient, and isometric to a Euclidean sphere". Next, we obtain the result: A complete almost Ricci soliton whose metric $g$ is $K$-contact and flow vector field $X$ is contact becomes Ricci soliton with constant scalar curvature. In particular, for $X$ strict, $g$ becomes compact Sasakian Einstein. Finally, we show that the Lie-bracket of two distinct Ricci soliton vector fields with the same metric generates a steady Ricci soliton.

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*Keywords*: Almost Ricci soliton, Conformal vector field, Constant scalar curvature, $K$-contact metric, Einstein Sasakian metric.

## 1 Introduction

Modifying the Ricci soliton equation by allowing the dilation constant $\lambda$ to become a variable function, Pigola et al. [8] defined an almost Ricci soliton as a Riemannian manifold $(M, g)$ satisfying the condition:

$$\pounds_X g_{ij} + 2R_{ij} = 2\lambda g_{ij}. \tag{1}$$

where $X$ is a vector field on $M$, $g_{ij}$ and $R_{ij}$ are the components of the metric tensor $g$ and its Ricci tensor in local coordinates $(x^i)$, $\pounds_X$ is the Lie-derivative operator along $X$, and $\lambda$ is a smooth function on $M$. A simple example is the canonical metric $g$ on a Euclidean sphere with $X$ a non-homothetic conformal vector field. For $\lambda$ constant, (1) becomes the Ricci soliton. The almost Ricci soliton is said to be shrinking, steady, and expanding according as $\lambda$ is positive, zero, and negative respectively; otherwise is indefinite. If the vector field $X$ is the gradient of a smooth function $f$, upto the addition of a Killing vector field, $(M, g, X, \lambda)$ is called a gradient almost Ricci soliton, in which case the equation (1) assumes the form:

$$\nabla_i \nabla_j f + R_{ij} = \lambda g_{ij}. \tag{2}$$

For an almost Ricci soliton with $X$ homothetic, $g$ is Einstein and hence $\lambda$ becomes constant and it becomes the trivial Ricci soliton. For $X$ non-homothetic, $g$ is a non-trivial almost Ricci soliton. We also note for an almost Ricci soliton that $X$ is conformal if and only if $g$ is Einstein.

Ricci solitons are special solutions of the Ricci flow equation

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}(t), \tag{3}$$

of the form $g_{ij}(t) = \sigma(t)\psi_t^* g_{ij}$ with initial condition $g_{ij}(0) = g_{ij}$, where $\psi_t$ are diffeomorphisms of $M$ and $\sigma(t)$ is the scaling function. In the same vein, we can view almost Ricci soliton as a special solution of Ricci flow, by considering the ansatz:

$$g_{ij}(t) = \sigma(t, x^k)\psi_t^* g_{ij}, \tag{4}$$

where $\psi_t$ are diffeomorphisms of $M$ generated by the family of vector fields $Y(t)$, and $\sigma(t, x^k)$ can be viewed as a pointwise scaling function that depends not only on time $t$, but also on the coordinates $x^k$ of points. The initial conditions: $g_{ij}(0) = g_{ij}$, $\psi_0 =$ identity, imply $\sigma(0, x^k) = 1$. Differentiating (4) with respect to $t$, using the Ricci flow equation (3), and substituting $t = 0$ shows

$$-2R_{ij} = (\frac{\partial}{\partial t}\sigma(t, x^k))|_{t=0} g_{ij} + \pounds_{Y(0)} g_{ij},$$

Labelling $Y(0)$ as $X$ and the time-independent function $(\frac{\partial}{\partial t}\sigma(t, x^k))|_{t=0}$ as $-2\lambda$, we obtain the almost Ricci soliton equation (1).

# 2 Compact Almost Ricci Soliton

It is well known that a compact Ricci soliton is gradient. This need not be true for almost Ricci soliton. In [3], Barros and Ribeiro Jr. showed that a compact gradient almost Ricci soliton with non-trivial conformal vector field is isometric to a Euclidean sphere. Intrigued by the fact that a compact Ricci soliton with constant scalar curvature is trivial (i.e. $X$ is Killing and $g$ is Einstein), Barros, Batista and Ribeiro Jr. [2] proved the following result.

**Theorem 1 (B-B-R)** *Let $(M^n, g, X, \lambda)$ be a compact oriented almost Ricci soliton. If Ric, $S$ and $dv_g$ denote respectively the Ricci tensor, scalar curvature and the volume form with respect to $g$, then*

$$\int_M |Ric - \frac{S}{n} g|^2 dv_g = \frac{n-2}{2n} \int_M g(\nabla S, X) dv_g. \tag{5}$$

*If, in addition; $n > 2$, the almost Ricci soliton is non-trivial and the scalar curvature is constant, then $(M, g)$ is isometric to a Euclidean sphere and the almost Ricci soliton is gradient.*

In this paper we provide a short Lie-derivative theoretic proof of this result, based on equations of evolution of Christoffel symbols and curvature quantities along the flow vector field $X$. We denote the Levi-Civita connection, connection coefficients, and components of curvature tensor of $g$ by $\nabla$, $\Gamma^i_{jk}$, and $R^h_{kji}$ respectively.

**Another Proof Of Theorem 1 (B-B-R).** Let us denote the inverse of $g_{ij}$ by $g^{ij}$. Taking the Lie-derivative of the relation $g_{ij} g^{jk} = \delta^k_i$ along $X$, using equation (1) and subsequently operating the resulting equation by $g^{il}$ we immediately get

$$\pounds_X g^{kl} = 2R^{kl} - 2\lambda g^{kl}. \tag{6}$$

Next, the use of equation (1) in the formula (page 23, Yano [9]):

$$\pounds_X \Gamma^h_{ij} = \frac{1}{2} g^{ht} [\nabla_j (\pounds_X g_{it}) + \nabla_i (\pounds_X g_{jt}) - \nabla_t (\pounds_X g_{ij})],$$

yields the evolution equation

$$\begin{aligned} \pounds_X \Gamma^h_{ij} &= \nabla^h R_{ij} - \nabla_j R^h_i - \nabla_i R^h_j - (\nabla^h \lambda) g_{ij} \\ &+ (\nabla_j \lambda)\delta^h_i + (\nabla_i \lambda)\delta^h_j. \end{aligned} \tag{7}$$

Let us follow the notational convention: $\nabla_k \nabla_j Z^h - \nabla_j \nabla_k Z^h = R^h_{kji} Z^i$, where $Z^i$ are components of an arbitrary vector field, and $R^k_{kji} = R_{ji}$. Using equation (7) in the following commutation formula (page 23, [9]):

$$\nabla_k (\pounds_X \Gamma^h_{ij}) - \nabla_j (\pounds_X \Gamma^h_{ik}) = \pounds_X R^h_{kji},$$

we obtain the evolution equation:

$$\begin{aligned}
\pounds_X R^h_{kji} &= \nabla_j\nabla_k R^h_i - \nabla_k\nabla_j R^h_i + \nabla_j\nabla_i R^h_k - \nabla_k\nabla_i R^h_j \\
&+ \nabla_k\nabla^h R_{ij} - \nabla_j\nabla^h R_{ik} + (\nabla_k\nabla_i\lambda)\delta^h_j \\
&- (\nabla_k\nabla^h\lambda)g_{ij} - (\nabla_i\nabla_j\lambda)\delta^h_k + (\nabla_j\nabla^h\lambda)g_{ik}.
\end{aligned}$$

Contracting this equation with $g^{hk}$ and using the twice contracted Bianchi identity: $\nabla_i R^i_j = \frac{1}{2}\nabla_j S$, we have

$$\begin{aligned}
\pounds_X R_{ji} &= \nabla_j\nabla_i S - \nabla_h\nabla_j R^h_i - \nabla_h\nabla_i R^h_j \\
&+ \Delta R_{ij} - (\Delta\lambda)g_{ij} - (n-2)\nabla_i\nabla_j\lambda.
\end{aligned}$$

Lie-differentiating $S = R_{ij}g^{ij}$ along $X$, and using the above equation and equation (6) provides the evolution equation for the scalar curvature:

$$\pounds_X S = 2R_{ij}R^{ij} + \Delta S - 2\lambda S - 2(n-1)\Delta\lambda. \tag{8}$$

Writing $\pounds_X S$ as $g(\nabla S, X)$, integrating the above equation over the compact $M$ and using the Gauss divergence theorem we get

$$\int_M [R_{ij}R^{ij} - \lambda S - \frac{1}{2}g(\nabla S, X)]dv_g = 0. \tag{9}$$

At this point, we note

$$div(SX) = \nabla_i(SX^i) = g(\nabla S, X) + S div X,$$

and integrate it over $M$ in order to get

$$\int_M [g(\nabla S, X) + S div X]dv_g = 0. \tag{10}$$

Now we contract equation (1) with $g^{ij}$ in order to get $divX = n\lambda - S$, and use it in (10) to obtain

$$\int_M (n\lambda S - S^2 + g(\nabla S, X))dv_g = 0.$$

Eliminating $\int_M (\lambda S)dv_g$ between the above equation and (9) and noting $|Ric - \frac{S}{n}g|^2 = R_{ij}R^{ij} - \frac{S^2}{n}$ we obtain equation (5), proving the first part of the theorem.

For the second part, we use the hypothesis that $S$ is constant in equation (5) to conclude that $g$ is Einstein. Thus, equation (1) reduces to $\pounds_X g_{ij} = 2(\lambda - \frac{S}{n})g_{ij}$, i.e. $X$ is a non-homothetic conformal vector field on $M$. With the setting $\lambda - \frac{S}{n} = \rho$, the foregoing conformal equation assumes the form

$$\pounds_X g_{ij} = 2\rho g_{ij}. \tag{11}$$

Using the conformal integrability condition (p. 26, [9])

$$\pounds_X R_{ij} = (2-n)\nabla_i \nabla_j \rho - (\Delta\rho) g_{ij}$$

and the Einstein condition $R_{ij} = \frac{S}{n} g_{ij}$ we get

$$(\Delta\rho + \frac{2S}{n}\rho) g_{ij} = (2-n)\nabla_i \nabla_j \rho. \tag{12}$$

Contracting it with $g^{ij}$ gives $\Delta\rho = -\frac{S}{n-1}\rho$. Using this in the identity: $\Delta\rho^2 = \nabla^i \nabla_i(\rho^2) = 2[|\nabla\rho|^2 + \rho\Delta\rho]$, and integrating over $M$ gives $\int_M |\nabla\rho|^2 = \frac{S}{n-1}\int_M \rho^2$. This shows that $S > 0$. Consequently, equation (12) becomes

$$\nabla_i \nabla_j \rho = -\frac{S}{n(n-1)}\rho g_{ij}. \tag{13}$$

This implies, by virtue of Obata's theorem [7]: "A complete Riemannian manifold $(M, g)$ of dimension $n \geq 2$ admits a non-trivial solution $\rho$ of the system of partial differential equations $\nabla_i \nabla_j \rho = -c^2 \rho g_{ij}$ ($c$ a positive constant) if and only if $M$ is isometric to a Euclidean sphere of radius $1/c$" that $(M, g)$ is isometric to a Euclidean sphere of radius $\sqrt{\frac{n(n-1)}{S}}$.

Equation (13) can also be expressed as $\pounds_{\nabla\rho} g_{ij} = \frac{2S}{n(1-n)}\rho g_{ij}$. Combining this with (11) we obtain

$$\pounds_{X - \frac{n(n-1)}{S}\nabla\rho} g_{ij} = 0.$$

Hence $X = \nabla(\frac{n(1-n)}{S}\rho)$ + a Killing vector field, i.e. the almost Ricci soliton is gradient, completing the proof.

## 3 $K$-Contact Metric As Almost Ricci Soliton

A $(2m+1)$-dimensional smooth manifold $M$ is called a contact manifold if it carries a global 1-form $\eta$ such that $\eta \wedge (d\eta)^m \neq 0$ everywhere on $M$. For a given contact 1-form $\eta$ there exists a unique vector field $\xi$ (Reeb vector field) such that $(d\eta)(\xi, .) = 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact subbundle $\eta = 0$, one obtains a Riemannian metric $g$ and a (1,1)-tensor field $\varphi$ such that

$$(d\eta)(Y,Z) = g(Y, \varphi Z), \eta(Z) = g(\xi, Z), \varphi^2 = -I + \eta \otimes \xi, \tag{14}$$

for arbitrary vector fields $Y, Z$ on $M$. We call $g$ an associated metric of $\eta$ and $(\varphi, \eta, \xi, g)$ a contact metric structure. A $K$-contact metric is a contact metric for which $\xi$ is Killing, equivalently:

$$Ric(\xi, Y) = 2mg(\xi, Y), \tag{15}$$

for an arbitrary vector field $Y$ on $M$. This condition is also equivalent to:

$$Ric(\xi, \xi) = 2m. \tag{16}$$

For details we refer to [4]. A contact metric $g$ on $M^{2m+1}$ is called Sasakian if the almost Kaehler structure induced on the cone $(\mathcal{R}^+ \times M)$ with metric $dr^2 + r^2 g$, is Kaehler (see Boyer and Galicki [5]). A Sasakian metric is $K$-contact, but the converse need not be true, except in dimension 3.

We would like to consider an almost Ricci soliton $(M, g, X, \lambda)$ such that $g$ is a $K$-contact metric and $X$ is a contact vector field. Let us recall that a vector field $X$ on a contact manifold is said to be a contact vector field if

$$\pounds_X \eta = f\eta, \tag{17}$$

for a smooth function $f$ on $M$. The contact vector field $X$ is called strict when $f = 0$.

Using Cartan's magic formula, we find that $\pounds_\xi \eta = d i_\xi \eta + i_\xi d\eta = d(1) + d\eta(\xi, .) = 0$, i.e. $\xi$ is a strict contact vector field. We note from equation (1) that, if we take $g$ as a $K$-contact metric and $X$ as $\xi$, then (as $\xi$ is Killing), the $K$-contact metric $g$ reduces to an Einstein metric and $\lambda$ becomes constant, equal to the Einstein constant $2m$, as seen from equation (16). We generalize this special situation in the form of the following result.

**Theorem 2** *Let $(M, g, X, \lambda)$ be a complete almost Ricci soliton with $g$ a $K$-contact metric and $X$ a contact vector field. Then it becomes Ricci soliton and $g$ has constant scalar curvature. In particular, if $X$ is strict, then $g$ is Sasakian Einstein.*

**Proof.** First of all, we have, by definition of the contact structure, $\omega = \eta \wedge (d\eta)^m \neq 0$ and thus is a volume element. Denote it by $\omega$. Using the hypothesis (17) we compute $\pounds_X d\eta = d\pounds_X \eta = (df)\wedge\eta + f(d\eta)$. Consequently, the formula: $\pounds_X \omega = (divX)\omega$ yields the relation $divX = (m+1)f$. On the other hand, the trace of equation (1) is $divX = (2m+1)\lambda - S$. Comparing the two values of $divX$ we have

$$S = (2m+1)\lambda - (m+1)f. \tag{18}$$

Next, we Lie-differentiate the second equation in (14) along $X$, and then use equations (1), (15) and (17) in order to get

$$\pounds_X \xi = (f - 2\lambda + 4m)\xi. \tag{19}$$

The Lie-derivative of $g(\xi, \xi) = 1$ (as $\xi$ is unit) along $X$, and the use of equations (1) and (16) provides $g(\pounds_X \xi, \xi) = 2m - \lambda$. The inner product of (19) with $\xi$ and the foregoing equation lead us to the relation: $f = \lambda - 2m$. Consequently, we have

$$\pounds_X \eta = (\lambda - 2m)\eta, \quad \pounds_X \xi = (2m - \lambda)\xi. \tag{20}$$

At this point, we take the Lie-derivative of the first equation in (14), along $X$ and use equations (1) and (17) in order to obtain

$$\eta(Z)\nabla f - (Zf)\xi + 2(f - 2\lambda)\varphi Z = -4Q\varphi Z + 2(\pounds_X \varphi)Z, \tag{21}$$

where $Z$ is an arbitrary vector field on $M$, and $Q$ is the Ricci tensor of type (1,1), defined by $g(Q., .) = Ric(., .)$. Substituting $\xi$ for $Z$ in equation (21) and using the property $\varphi\xi = 0$ and equation (20) we find $\nabla f = (\xi f)\xi$, i.e. $df = (\xi f)\eta$. Taking its exterior derivative, using Poincare lemma: $d^2 = 0$, and then wedge product with $\eta$ we have $(\xi f)\eta \wedge d\eta = 0$. As $\eta \wedge d\eta$ cannot vanish anywhere, otherwise the definition of the contact structure would be violated, we conclude that $\xi f = 0$, and hence $df = 0$, i.e. $f$ is constant on $M$. Consequently, equation (21) reduces to the following evolution equation for $\varphi$:

$$\pounds_X \varphi = 2Q\varphi - (2m + \lambda)\varphi. \tag{22}$$

As shown earlier, $f = \lambda - 2m$, and $f$ is constant, we conclude that $\lambda$ is constant and hence the almost Ricci soliton becomes Ricci soliton. Appealing to equation (18), we find that $S$ is constant. This proves first part. For the second part, the hypothesis $f = 0$ immediately implies $\lambda = 2m$ and thus we get from (18) that $S = 2m(2m+1)$. Plugging these findings in equation (8) and carrying out a straightforward computation shows $|Ric - 2mg|^2 = 0$. Hence $Ric = 2mg$, i.e. $g$ is Einstein with Einstein constant $2m$.

As $(M, g)$ is complete, thanks to Myers' theorem, $(M, g)$ becomes compact. In order to turn $g$ into Sasakian, we recall the following result of Morimoto [6]: "Let $(M, \eta, g)$ be a compact $K$-contact manifold such that $g$ is $\eta$-Einstein, i.e. its Ricci tensor satisfies $Ric = ag + b\eta \otimes \eta$ for real constants $a, b$. If $a > -2$, then $g$ is Sasakian". This result was also proved independently by Boyer and Galicki [5], and Apostolov et al. [1]. In our case, $a = 2m$ and $b = 0$, and hence the aforementioned result holds. Thus, we conclude that $g$ is Sasakian, and complete the proof.

# 4 Commutation Of Ricci Soliton Vector Fields

We consider two distinct Ricci solitons with the same Riemannian metric and show that the Lie-bracket of their flow vector fields give rise to a steady Ricci soliton. More precisely, we prove

**Proposition 1** *Let $(M, g, X_1, \lambda_1)$ and $(M, g, X_2, \lambda_2)$ be two distinct non-trivial Ricci solitons. Then, $[X_1, X_2]$ determines a steady Ricci soliton on $M$ with a metric homothetic to $g$.*

**Proof** By hypothesis, we have

$$\pounds_{X_1} g + 2Ric = 2\lambda_1 g, \quad \pounds_{X_2} g + 2Ric = 2\lambda_2 g, \tag{23}$$

where $\lambda_1$ and $\lambda_2$ are constants. As these are two distinct Ricci solitons, we may assume without any loss of generality, that $\lambda_1 < \lambda_2$. The two equations in (23) show that $X_1 = X_2 + H$ where $H$ is a homothetic vector field satisfying $\pounds_H g = 2(\lambda_1 - \lambda_2)g$. The following computation:

$$\begin{aligned} \pounds_{[X_1, X_2]} g &= \pounds_{[X_2 + H, X_2]} g = \pounds_{[H, X_2]} g = \pounds_H \pounds_{X_2} g - \pounds_{X_2} \pounds_H g \\ &= \pounds_H(-2Ric + 2\lambda_2 g) - \pounds_{X_2}(2(\lambda_1 - \lambda_2)g) = 4(\lambda_1 - \lambda_2)Ric, \end{aligned}$$

shows that

$$\pounds_{\frac{1}{2(\lambda_2-\lambda_1)}[X_1,X_2]}g + 2Ric = 0.$$

Taking into account the fact that the Ricci tensor is invariant under a homothetic transformation and noticing that $[X_1, X_2]$ cannot be conformal (otherwise $g$ would become Einstein) we conclude that $(M, \frac{1}{2(\lambda_2-\lambda_1)}g, [X_1, X_2], 0)$ is a Ricci soliton which is steady. This completes the proof.

# 5 Concluding Remarks

1. In the proof of Theorem 1, Barros, Batista and Ribiero Jr. used a result of Yano and Nagano and the Hodge-de Rham decomposition. Our proof uses a theorem of Obata and does not need Hodge-de Rham decomposition.

2. The hypotheses of Theorem 2 can be interpreted in terms of contact Hamiltonians as follows. The contact Hamiltonian associated to a contact vector field $X$ defined by equation (17) is a function $\mathcal{H}$ defined as $\eta(X)$, and the function $f$ turns out to be equal to $\xi\mathcal{H}$. The vector field $X$ is the Hamiltonian vector field associated to $\mathcal{H}$. Computing $\pounds_X\mathcal{H} = \pounds_X(\eta(X)) = (\pounds_X\eta)X = f\eta(X) = f\mathcal{H} = (\xi\mathcal{H})\mathcal{H}$ shows that the contact vector field $X$ is strict, i.e. $f = \xi\mathcal{H} = 0$ if and only if the associated Hamiltonian $\mathcal{H}$ is a first integral of $X$, i.e. is preserved along the flow of the Hamiltonian vector field $X$.

3. For the second part of Theorem 2, we found that $\lambda = 2m$, $Ric = 2mg$ and hence $Q = 2mI$. Using these and the hypothesis $f = 0$ in equations (20) and (22) we infer that $X$ preserves all structure tensors $\eta, \xi, g, \varphi$, and hence is an infinitesimal automorphism of the Sasakian structure on $M$.

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